Separability Conditions in Acts over Monoids

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Finiteness conditions

A **finiteness condition** for a class of algebraic structures is a property that is satisfied by at least all finite members of that class.

This talk is concerned with *separability conditions* for the class of *monoid acts*.

An algebra A is **residually finite** if for any pair of distinct elements $a, b \in A$ there exist a finite algebra B and a homomorphism $\theta : A \to B$ such that $a\theta \neq b\theta$.

For any finitely presented, residually finite algebra, the word problem is solvable.

Separability

Let $\mathcal{K}$ be a class of algebras, let $A \in \mathcal{K}$, and let $\mathcal{S}$ be a collection of non-empty subsets of A. We say that A satisfies the **separability condition with respect to** $\mathcal{S}$ if for any $X \in \mathcal{S}$ and any $a \in A \setminus X$, there exist a finite algebra $B \in \mathcal{K}$ and a homomorphism $\theta : A \to B$ such that $a\theta \notin X\theta$.

Residual finiteness may be viewed as the separability condition with respect to the collection of all singleton subsets.

An algebra A is:

- **weakly subalgebra separable** if it satisfies the separability condition with respect to the collection of all finitely generated subalgebras;
- **strongly subalgebra separable** if it satisfies the separability condition with respect to the collection of all subalgebras;
- **completely separable** if it satisfies the separability condition with respect to the collection of all non-empty subsets.

Separability

Proposition. For an algebra A the following statements hold.

- If A is completely separable then it is strongly subalgebra separable.
- If A is strongly subalgebra separable then it is weakly subalgebra separable.
- If A is completely separable then it is residually finite.

Lemma. Let A be an algebra, let B be a subalgebra of A, and let $\mathcal{C}$ be any of the following separability conditions: residual finiteness, weak subalgbra separability, strong subalgebra separability, complete separability. If A satisfies $\mathcal{C}$ then so does B.

M-acts

Let M be a monoid with identity 1. A (**right**) M-**act** is a non-empty set A together with a map

$$A \times M \to A, (a, m) \mapsto am$$

such that $a(mn) = (am)n$ and $a1 = a$ for all $a \in A$ and $m, n \in M$.
For instance, M itself is an M-act via right multiplication; for clarity, we denote it by $\mathbf{M}$.

Given a non-empty set X, the *full transformation monoid* on X is the set of all transformations of X under composition of mappings, denoted by T_X.

Proposition. Given an M-act A, we obtain a monoid homomorphism $\theta : M \to T_A$ by defining $a(m\theta) = am$ for all $a \in A$ and $m \in M$. Conversely, given a monoid homomorphism $\theta : M \to T_A$, the set A is turned into an M-act by defining $am = a(m\theta)$ for all $a \in A$ and $m \in M$.

Basic definitions and facts

A non-empty subset B of an M-act A is a **subact** of A if $bm \in B$ for all $b \in B, m \in M$.
The subacts of **M** are precisely the right ideals of M.

Let A and B be M-acts. A map $\theta : A \to B$ is an M-**homomorphism** if $(am)\theta = (a\theta)m$ for all $a \in A, m \in M$.

A subset U of an M-act A is a **generating set** for A if $A = UM = \{um : u \in U, m \in M\}$.

An M-act A is said to be **finitely generated** (resp. **cyclic**) if it has a finite (resp. one element) generating set.

Basic definitions and facts

An equivalence relation ρ on an M-act A is an **congruence** on A if $(a,b) \in \rho$ implies $(am, bm) \in \rho$ for all $a, b \in A, m \in M$.

For a congruence ρ on A, the quotient set $A/\rho = \{[a]_\rho : a \in A\}$ becomes an M-act by defining $[a]_\rho m = [am]_\rho$ for all $a \in A, m \in M$.

Proposition. An M-act A is cyclic if and only if there exists a right congruence ρ on M such that $A \cong \mathbf{M}/\rho$.

Given a subact B of A, define ρ_B on A by $(a,b) \in \rho_B \Leftrightarrow a = b$ or $a, b \in B$. The quotient A/ρ_B is denoted by A/B and called the **Rees quotient** of A by B.
We will identify the ρ_B-class $\{a\} \in A/B$ with a, and denote the ρ_B-class $B \in A/B$ by 0_B.

Separability in acts

- An M-act A is **residually finite** (RF) if for any pair of distinct elements $a, b \in A$, there exist a finite M-act B and an M-homomorphism $\theta : A \to B$ such that $a\theta \neq b\theta$.
- An M-act A is **weakly subact separable** (WSS) if for any finitely generated subact B of A and any $a \in A \setminus B$, there exist a finite M-act C and an M-homomorphism $\theta : A \to C$ such that $a\theta \notin B\theta$.
- An M-act A is **strongly subact separable** (SSS) if for any subact B of A and any $a \in A \setminus B$, there exist a finite M-act C and an M-homomorphism $\theta : A \to C$ such that $a\theta \notin B\theta$.
- An M-act A is **completely separable** (CS) if for any non-empty subset $X \subseteq A$ and any $a \in A \setminus X$, there exist a finite M-act B and an M-homomorphism $\theta : A \to B$ such that $a\theta \notin X\theta$.

Proposition. An M-act A is WSS if and only if it is cyclic subact separable.

Separability in certain cyclic acts

Theorem. Let M be a monoid and let ρ be a congruence on M.
(1) The M-act $\mathbf{M}/\rho$ is RF $\Leftrightarrow$ the monoid M/ρ is RF (as a monoid).
(2) The M-act $\mathbf{M}/\rho$ is WSS $\Leftrightarrow$ the monoid M/ρ satisfies the separability condition with respect to the collection of all principal right ideals.
(3) The M-act $\mathbf{M}/\rho$ is SSS $\Leftrightarrow$ the monoid M/ρ satisfies the separability condition with respect to the collection of all right ideals.
(4) The M-act $\mathbf{M}/\rho$ is CS $\Leftrightarrow$ the monoid M/ρ is CS.

Corollary. Let M be a monoid. If every cyclic M-act is RF (resp. CS), then every quotient of M is RF (resp. CS).

Separability in certain cyclic acts

Corollary. Let M be a monoid.
(1) $\mathbf{M}$ is RF $\Leftrightarrow$ M is RF.
(2) $\mathbf{M}$ is WSS $\Leftrightarrow$ M satisfies the separability condition with respect to the collection of all principal right ideals.
(3) $\mathbf{M}$ is SSS $\Leftrightarrow$ M satisfies the separability condition with respect to the collection of all right ideals.
(4) $\mathbf{M}$ is CS $\Leftrightarrow$ M is CS.

Corollary. Let G be a group.
(1) $\mathbf{G}$ is RF if and only if G is RF (as a group).
(2) $\mathbf{G}$ *is* SSS (and hence WSS).
(3) $\mathbf{G}$ is CS if and only if G is finite.

Finiteness conditions on monoids

For each of the four separability conditions $\mathcal{C}$, we investigate:

- which monoids have the property that all their acts satisfy $\mathcal{C}$;
- which monoids have the property that all their finitely generated acts satisfy $\mathcal{C}$.

Given an M-act A, for each pair $a, b \in A$ define a set

$$S(b,a) = \{m \in M : bm = a\} \subseteq M.$$

Theorem. An M-act A is CS $\Leftrightarrow$ for each $a \in A$, $\{S(b,a) : b \in A\}$ is finite.

Corollary. If M is a finite monoid, then every M-act is CS.

Proposition. If M is a monoid with finitely many $\mathcal{R}$-classes, then all M-acts are SSS.

Groups

Corollary. Let G be a group.

- All G-acts are CS $\Leftrightarrow$ G is finite.
- All G-acts *are* SSS.

Theorem. The following are equivalent for a group G:

1. all G-acts are RF;
2. all finitely generated G-acts are RF;
3. G is strongly *subgroup* separable.

Corollary. If G is a polycyclic-by-finite group, then all G-acts are RF.

Corollary. The following are equivalent for a nilpotent group G:

1. all G-acts are RF;
2. for every normal subgroup N of G, the quotient G/N is RF.

Equivalent formulations

Proposition. The following are equivalent for a monoid M:

1. all M-acts are SSS;
2. all M-acts are WSS;
3. for any M-act A containing zeroes, any zero $0 \in A$ and any $a \in A \setminus \{0\}$, there exist a finite M-act C and an M-homomorphism $\theta : A \to C$ such that $a\theta \neq 0\theta$.

Corollary. If all M-acts are RF, then all M-acts are SSS.

Equivalent formulations

Proposition. The following are equivalent for a monoid M:

1. all finitely generated M-acts are SSS;
2. all finitely generated M-acts are WSS;
3. for any finitely generated M-act A containing zeroes, any zero $0 \in A$ and any $a \in A \setminus \{0\}$, there exist a finite M-act C and an M-homomorphism $\theta : A \to C$ such that $a\theta \neq 0\theta$.

Corollary. If all finitely generated M-acts are RF, then all finitely generated M-acts are SSS.

Equivalent formulations

Theorem. For each separability condition $\mathcal{C}$, all finitely generated M-acts satisfy $\mathcal{C}$ if and only if all cyclic M-acts satisfy $\mathcal{C}$.

Corollary. The following are equivalent for a monoid M:

1. all finitely generated M-acts are RF;
2. every right congruence on M is the intersection of a family of finite index right congruences on M.

Corollary. Let M be a monoid for which every right congruence is a (two-sided) congruence. Then the following are equivalent:

1. all finitely generated M-acts are RF (resp. CS);
2. every quotient of M is RF (resp. CS).

Finitely generated commutative monoids

Theorem. Let M be a finitely generated commutative monoid. Then all finitely generated M-acts are RF and SSS.

Theorem. TFAE for a finitely generated commutative monoid M:

1. all finitely generated M-acts are CS;
2. for every congruence ρ on M, every $\mathcal{H}$-class of the quotient M/ρ is finite.

Corollary. All finitely generated $\mathbb{N}_0$-acts are CS.

Example. Let $A = \{a_i : i \in \mathbb{N}_0\} \cup \{0\}$, and define an action of $\mathbb{N}_0$ on A by

$$a_i \cdot j = \begin{cases} a_{i-j} & \text{if } i \geq j, \\ 0 & \text{otherwise,} \end{cases} \qquad 0 \cdot j = 0.$$

The $\mathbb{N}_0$-act A is not RF.

Clifford monoids

A *Clifford monoid* is an inverse monoid whose idempotents are central.

Theorem. The following are equivalent for a Clifford monoid M:

1. all M-acts are CS;
2. all finitely generated M-acts are CS;
3. M is finite.

Theorem. Let M be a Clifford monoid. Then all M-acts are SSS.

Theorem. Let M be a commutative idempotent monoid. Then all M-acts are RF and SSS.

Clifford monoids

Proposition. Let M be a Clifford monoid. If all finitely generated M-acts are RF, then every maximal subgroup of M is strongly *subgroup* separable.

Theorem. Let M be a Clifford monoid, and let $M = \mathcal{S}(Y, G_\alpha)$ be its decomposition into a semilattice of groups. If every subsemilattice of Y has a least element, then the following are equivalent:

1. all M-acts are RF;
2. all finitely generated M-acts are RF;
3. for each $\alpha \in Y$, the group G_α is strongly subgroup separable.

Open Problems

Does there exist an infinite commutative monoid whose acts are all CS?

Let M be a Clifford monoid. If every maximal subgroup of M is strongly subgroup separable, are all (finitely generated) M-acts RF?